

UE5 Fundamentals of Algorithms

Lecture 4-5-6: Programming strategies

Ecole Centrale de Lyon, Bachelor of Science in Data Science for Responsible Business

Romain Vuillemot



Outline

- Definitions of programming strategies
- Divide and conquer
- Greedy algorithms
- Dynamic programming

Programming strategies

A programming strategy are algorithms aimed at solving a specific problem in a precise manner.

Examples of Strategies:

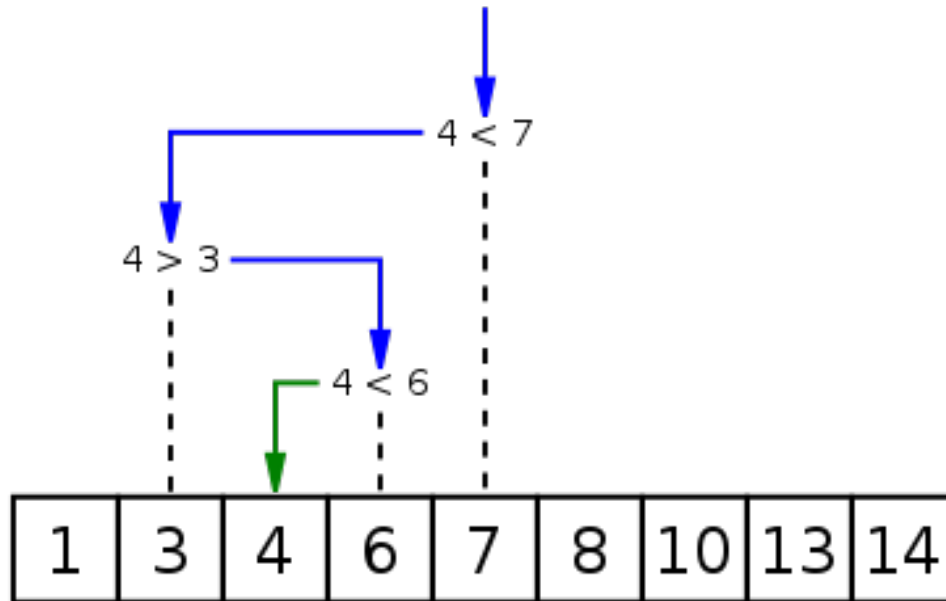
- **Divide and Conquer:** Divide a problem into simpler sub-problems, solve the sub-problems, and then combine the solutions to solve the original problem.
- **Dynamic Programming:** Solve a problem by breaking it down into sub-problems, calculating and memorizing the results of sub-problems to avoid unnecessary recomputation.
- **Greedy Algorithm:** Make a series of choices that seem locally optimal at each step to find a solution, with the hope that the result will be globally optimal as well.

Examples of divide and conquer algorithms:

- Binary search
- Quick sort and merge sort
- Map Reduce
- Others: Fast multiplication (Karatsuba)

Binary search

Given a sorted list, find or insert a specific value while keeping the order.

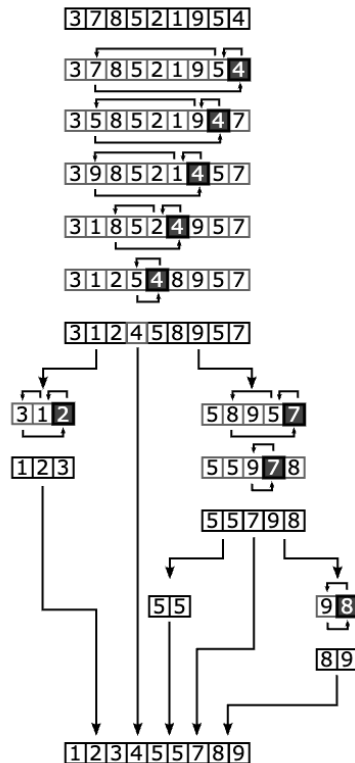


See [the notebook](#).

Quick sort

Recursive sorting algorithm which works in two steps:

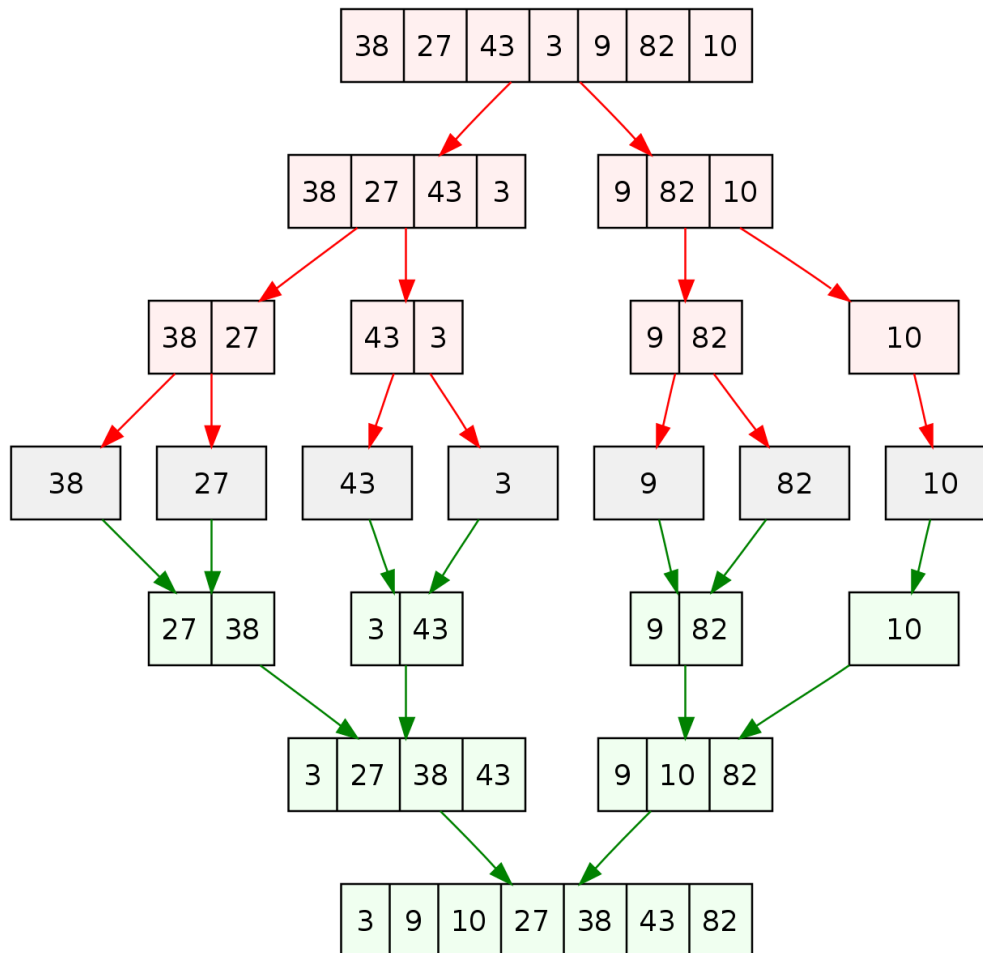
1. select a pivot element
2. partitioning the array into smaller sub-arrays, then sorting those sub-arrays.



Merge sort

Divide an array recursively into two halves (based on a *pivot* value), sorting each half, and then merging the sorted halves back together. This process continues until the entire array is sorted.

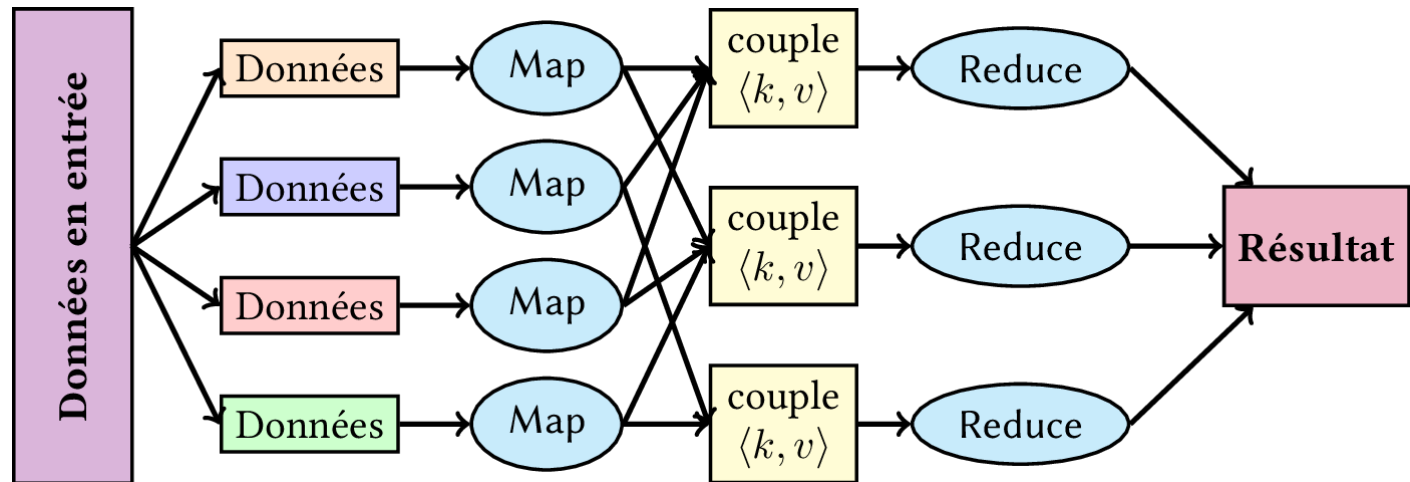
Complexity: $O(n\log(n))$.



Map reduce

Divide a large dataset into smaller chunks and processes them independantly. Two main steps:

- the Map stage, where data is filtered and transformed into key-value pairs
- the Reduce stage, where data is aggregated and the final result is produced.



Map reduce (without map reduce..)

Calculate the sum of squares values from a list of numerical values.

```
In [49]: data = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Map reduce (without map reduce..)

Calculate the sum of squares values from a list of numerical values.

```
In [49]: data = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

```
In [60]: result = {}
         for num in data:
             square = num * num
             result[square] = num

         final_result = list(result.items())

         print(final_result)
         print(sum([x[0] for x in final_result]))
```

```
[(1, 1), (4, 2), (9, 3), (16, 4), (25, 5), (36, 6), (49, 7), (64, 8), (81, 9), (100, 10)]
385
```

Map reduce (Python)

1. Divide the problem in sub-problems
2. Apply the mapping function
3. Reduce the results

Map reduce (Python)

1. Divide the problem in sub-problems
2. Apply the mapping function
3. Reduce the results

```
In [2]: data = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

def mapper(numbers):
    result = []
    for num in numbers: # calculate the squares
        result.append((num, num * num))
    return result

def reducer(pairs):
    result = {}
    for key, value in pairs: # sums the squares
        if key in result:
            result[key] += value
        else:
            result[key] = value
    return result.items()
```

Map reduce (Python)

1. Divide the problem in sub-problems
2. Apply the mapping function
3. Reduce the results

```
In [21]: chunk_size = 2
chunks = [data[i:i+chunk_size] for i in range(0, len(data), chunk_size)]

mapped_data = [mapper(chunk) for chunk in chunks]

grouped_data = {} # map
for chunk in mapped_data:
    for key, value in chunk:
        if key in grouped_data:
            grouped_data[key].append(value)
        else:
            grouped_data[key] = [value]

reduced_data = [reducer(list(grouped_data.items()))] # reduce
print(sum([x[1][0] for x in list(reduced_data[0])]))
```

Discussion on Divide and Conquer

- Similarities with recursion by dividing a problem in a sub-problem
- But with a combination step (which may hold most of the code difficulty)
- Can be implemented in a non-recursive way
- $n\log(n)$ complexity when split the problem and solves each split

Example: Change-making problem

$$Q_{opt}(S, M) = \min \sum_{i=1}^n x_i.$$

S : all the available coins

M : amount

Greedy solution:

1. Sort the coins in descending order
2. Initialize a variable to count coins used
3. Subtract the number of coins used (if limited)
4. Continue this process until amount becomes zero.

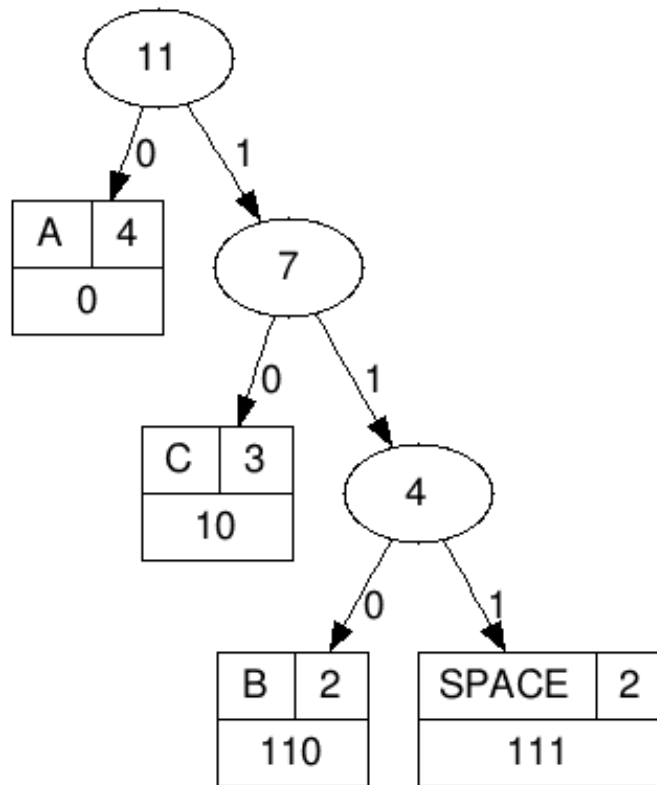
Example: Huffman code

A prefix code for lossless information compression. The key idea is to build a character frequency tree, and then traverse the tree to obtain the code (0 for left branches, 1 for right branches).

Steps to Follow:

1. Create a queue with letter frequencies.
2. Take the two least frequent characters from the list.
3. Create a node in a tree that contains the sum of frequencies and the two characters as children. Add this node to the queue and remove the children.
4. Repeat steps 2 and 3 until no more elements remain; the last element is the root of the tree.

Example: Huffman code



Example: Change-making problem (Python)

Greedy solution to return the minimal number of coins necessary.

```
In [82]: coins = [1, 2, 5]  
         amount = 11
```

Example: Change-making problem (Python)

Greedy solution to return the minimal number of coins necessary.

```
In [82]: coins = [1, 2, 5]
         amount = 11
```

```
In [83]: def coin_change_greedy(coins, amount):
         coins.sort(reverse=True) # important! sort in descending order

         coin_count = 0
         remaining_amount = amount

         for coin in coins:
             while remaining_amount >= coin:
                 remaining_amount -= coin
                 coin_count += 1

         if remaining_amount == 0:
             return coin_count
         else:
             return -1
```

```
In [81]: print(coin_change_greedy(coins, amount)) # 3 (11 = 5 + 5 + 1)
```

Example: Change-making problem (Python)

*Greedy solution that returns the **list of coins** used.*

Tip: use a list with the same structure as coins.

Example: Change-making problem (Python)

*Greedy solution that returns the **list of coins** used.*

Tip: use a list with the same structure as coins.


```
In [87]: def coin_change_greedy(coins, amount):
        coins.sort(reverse=True)

        coin_count = 0
        remaining_amount = amount
        used_coins = [0] * len(coins)

        for i, coin in enumerate(coins):
            while remaining_amount >= coin:
                remaining_amount -= coin
                coin_count += 1
                used_coins[i] += 1

        if remaining_amount == 0:
            return coin_count, used_coins
        else:
            return -1, []

        coins = [25, 10, 5, 1]
        amount = 63
        min_coins, coins_used = coin_change_greedy(coins, amount)

        print(f"Minimum coins needed: {min_coins}")
        print("Coins used:", coins_used)
```

Minimum coins needed: 6
Coins used: [2, 1, 0, 3]

Example: Change-making problem (Python)

*Greedy solution that returns the **list of coins** used from **a limited availability of coins**.*

Tip: use a list of coins availability of same structure as coins.

```
In [92]: coins = [25, 10, 5, 1]
amount = 63
coin_availability = [1, 2, 3, 4]
```


Example: Change-making problem (Python)

*Greedy solution that returns the **list of coins** used from **a limited availability of coins**.*

Tip: use a list of coins availability of same structure as coins.

```
In [92]: coins = [25, 10, 5, 1]
amount = 63
coin_availability = [1, 2, 3, 4]
```

```
In [91]: def coin_change_greedy(coins, amount, coin_availability):
    coins.sort(reverse=True)

    coin_count = 0
    remaining_amount = amount
    used_coins = [0] * len(coins)

    for i, coin in enumerate(coins):
        while remaining_amount >= coin and used_coins[i] < coin_availability:
            remaining_amount -= coin
            coin_count += 1
            used_coins[i] += 1

    if remaining_amount == 0:
        return coin_count, used_coins
    else:
        return -1, []

min_coins, coins_used = coin_change_greedy(coins, amount, coin_availability)

print(f"Minimum coins needed: {min_coins}")
print("Coins used:", coins_used)
```

```
Minimum coins needed: 9
Coins used: [1, 2, 3, 3]
```

```
In [ ]:
```

Discussion on Greedy algorithms

- Often considered as an *heuristic*
- Easy to understand, implement and communicate
- They often lead to non-optimal solution

Examples of dynamic programming algorithms

- Fibonacci Sequence
- Rod Cutting
- Sequence Alignment, Longest Subsequence Finding
- Shortest Path Finding

Fibonnacci (reminder)

To calculate the n -th number in the Fibonacci sequence, which is determined as follows:

latex Copy code $fib(n) = fib(n - 1) + fib(n - 2), n \in \mathbb{N}$ Where the sequence starts with 1, 1, and then continues as 2, 3, 5, 8, 13, 21, and so on, to find the 9th number ($n = 9$).

Let's calculate the 9th Fibonacci number step by step:

$$fib(1) = 1$$

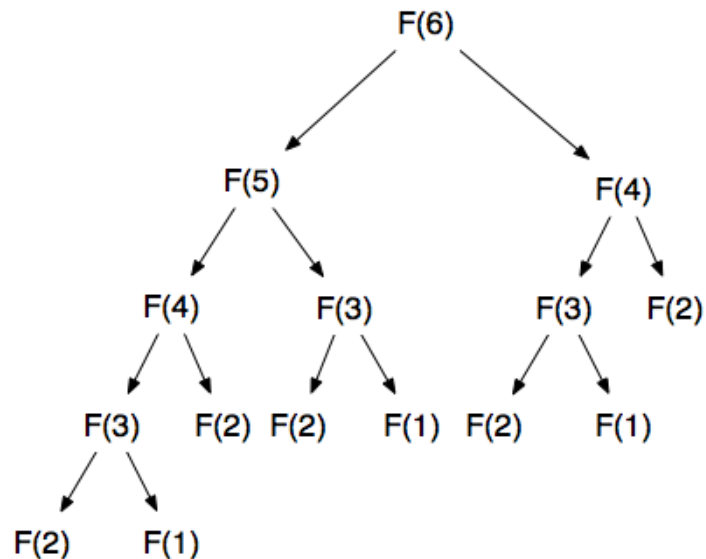
$$fib(2) = 1$$

$$fib(3) = fib(2) + fib(1) = 1 + 1 = 2$$

Fibonacci (naive)

```
In [4]: def fib(n):  
        if n < 2:  
            return n  
        else:  
            return fib(n - 1) + fib(n - 2)
```

Call tree (for $n = 6$):



Requires to calculate the same F-value multiple times.

Fibonacci (dynamic programming)

Optimized using a lookup table, which is a data structure to memoize values that have already been computed.

Fibonacci (dynamic programming)

Optimized using a lookup table, which is a data structure to memoize values that have already been computed.

```
In [97]: def fib(n, lookup):
    if n == 0 or n == 1:
        lookup[n] = n

    if lookup[n] is None:
        lookup[n] = fib(n - 1, lookup) + fib(n - 2, lookup)

    return lookup[n]

def main():
    n = 6

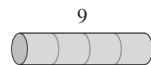
    lookup = [None] * (n + 1)
    result = fib(n, lookup)
    print(f"{n}-th Fibonacci number is {result}")

if __name__ == "__main__":
    main()
```

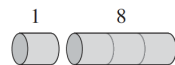
6-th Fibonacci number is 8

Rod cutting

Given a list of cuts and prices, identify the optimal cuts. Given the example below, what is the best cutting strategy for a rod of size 4 ?



(a)



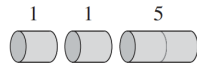
(b)



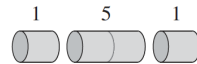
(c)



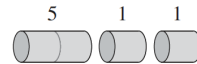
(d)



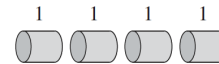
(e)



(f)



(g)

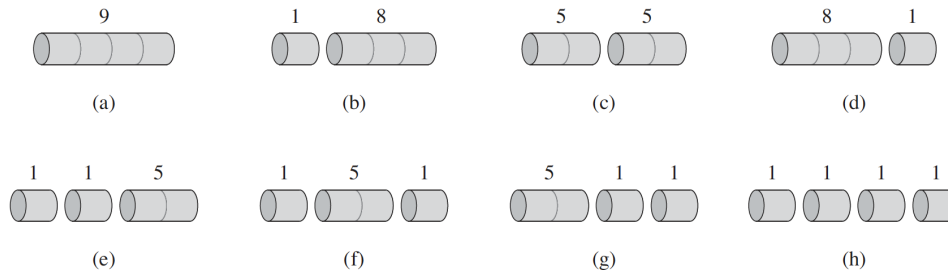


(h)

size (i)	1	2	3	4	5	6	7	8
price (p _i)	1	5	8	9	10	17	17	20

Rod cutting

Given a list of cuts and prices, identify the optimal cuts. Given the example below, what is the best cutting strategy for a rod of size 4 ?



size (i)	1	2	3	4	5	6	7	8
price (p _i)	1	5	8	9	10	17	17	20

Solution: For a rod of size 4 optimal solution is 2 cuts of size 2 so .

Rod cutting: check a solution

Given the previous table of size and price, check the cost of a given solution by defining a function `check_rod_cutting(prices, n)`.

Rod cutting: check a solution

Given the previous table of size and price, check the cost of a given solution by defining a function `check_rod_cutting(prices, n)`.

```
In [27]: def check_rod_cutting(prices, n):
          table = [0] * (n + 1)

          for i in range(1, n + 1):
              max_price = float('-inf')
              for j in range(1, i + 1):
                  max_price = max(max_price, prices[j] + table[i - j])
              table[i] = max_price

          return table[n]
```

```
In [30]: prices = [0, 1, 5, 8, 9, 10, 17, 17, 20]
          n = 2

          max_total_price = check_rod_cutting(prices, n)
          print(f"The maximum total price for a rod of length {n} is {max_total_p
```

The maximum total price for a rod of length 2 is 5

Rod cutting (brute force)

Let's solve the rod cutting problem using a brute force (naive) approach.

1. define a value function
2. identify a base case
3. identify a recursion mechanism

Rod cutting (brute force)

Let's solve the rod cutting problem using a brute force (naive) approach.

1. define a value function
2. identify a base case
3. identify a recursion mechanism

```
In [33]: def cut_brute_force(n, t):  
    if n == 0:  
        return 0  
    max_valeur = float('-inf')  
    for i in range(1, n + 1):  
        valeur_courante = t[i] + coupe_brute_force(n - i, t)  
        max_valeur = max(max_valeur, valeur_courante)  
    return max_valeur
```

Rod cutting (brute force)

Let's solve the rod cutting problem using a brute force (naive) approach.

1. define a value function
2. identify a base case
3. identify a recursion mechanism

```
In [33]: def cut_brute_force(n, t):  
        if n == 0:  
            return 0  
        max_valeur = float('-inf')  
        for i in range(1, n + 1):  
            valeur_courante = t[i] + coupe_brute_force(n - i, t)  
            max_valeur = max(max_valeur, valeur_courante)  
        return max_valeur
```

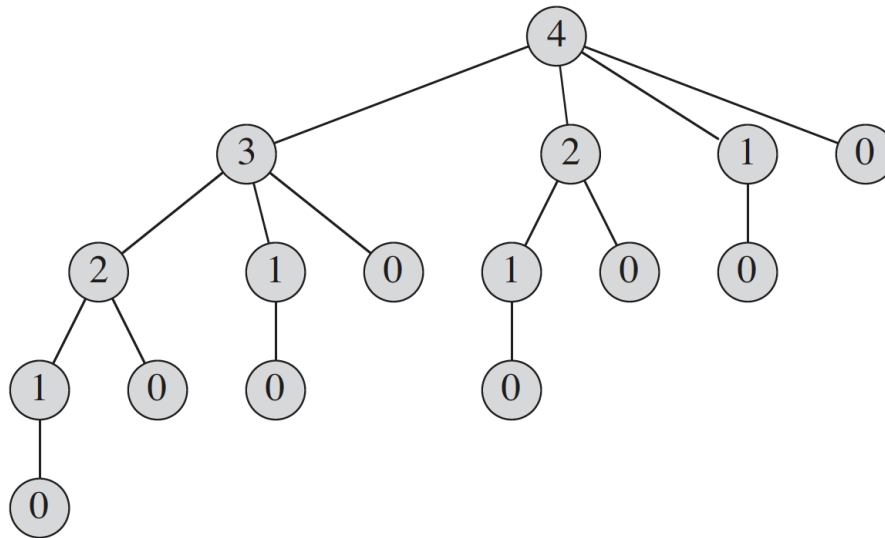
```
In [34]: lengths = [0, 1, 2, 3, 4, 5, 6, 7, 8]  
values = [0, 1, 5, 8, 9, 10, 17, 17, 20]  
rod_length = 2  
max_value = coupe_brute_force(rod_length, values)  
print(f"The maximum value for a rod of length {rod_length} is {max_value}")
```

The maximum value for a rod of length 2 is 5.

Rod cutting (dynamic programming)

General case:

- Cutting a rod of length i optimally.
- Cutting a rod of length $(n - i)$ optimally.



General case: $V_n = \max_{1 \leq i \leq n} (p_i + V_{n-i})$

size (i)	1	2	3	4	5	6	7	8
price (pi)	1	5	8	9	10	17	17	20

$$V_3 = \max_{1 \leq i \leq 3} (p_i + V_{3-i})$$

Let's calculate V_3 step by step for each possible value of i :

1. If $i = 1$, we cut the rod into two pieces: one of length 1 and one of length 2.

- $V_1 = p_1 = 2$
- $V_{3-1} = V_2$

2. If $i = 2$, we cut the rod into two pieces: one of length 2 and one of length 1.

- $V_2 = p_2 = 5$
- $V_{3-2} = V_1$

3. If $i = 3$, we cut the rod into one piece of length 3.

- $V_3 = p_3 = 9$
- $V_{3-3} = V_0$ (Assuming that $V_0 = 0$ as a base case.)

Now, we can calculate the values for V_2 and V_1 recursively using the same formula:

For V_2 :

$$V_2 = \max(p_1 + V_1, p_2 + V_0) = \max(2 + V_1, 5 + 0) = \max(2 + 2, 5 + 0) = \max(4, 5) = 5$$

For V_1 :

$$V_1 = \max(p_1 + V_0) = \max(2 + 0) = 2$$

So, V_2 is 5 and V_1 is 2.

Now, we can calculate V_3 using the values of V_2 and V_1 :

$$V_3 = \max(p_1 + V_2, p_2 + V_1, p_3 + V_0) = \max(1 + 5, 5 + 2, 9 + 0) = \max(6, 7, 9) = 9$$

Rod cutting (dynamic programming)

Rod cutting (dynamic programming)

```
In [36]: INT_MIN = 0

def cutRod(price, n):

    # init cache tables
    val = [0 for x in range(n+1)]
    val[0] = 0

    for i in range(1, n+1):
        max_val = INT_MIN
        for j in range(i):
            max_val = max(max_val, price[j] + val[i-j-1])
        val[i] = max_val

    return val[n]

if __name__=="__main__":
    arr = [1, 5, 8, 9, 10, 17, 17, 20]
    size = len(arr)
    print("Max size cut " + str(cutRod(arr, size)), len(arr) )
```

Max size cut 22 8

Change-making problem (dynamic programming)

$$Q_{opt}(S, M) = \min \sum_{i=1}^n x_i.$$

S : all the available coins

M : amount

$$Q_{opt}(i, m)$$

$$= \min \begin{cases} 1 + Q_{opt}(i, m - v_i) & \text{if } (m - v_i) \geq 0 & \text{we use a coin of type } i \text{ of value } v_i \\ Q_{opt}(i - 1, m) & \text{if } i \geq 1 & \text{we do not use coin of type } i, \text{ we use } i - 1 \end{cases}$$

	0	1	2	3	4	5	6	7
\emptyset	0	∞	∞	∞	∞	∞	∞	∞
1c	0	1	2	3	4	5	6	7
1c, 3c	0	1	2	1	2	3	2	3
1c, 3c, 4c	0	1	2	1	1	2	2	2

+1

The diagram illustrates the dynamic programming table for the change-making problem. The table has 5 rows and 9 columns. The columns represent the amount (0 to 7), and the rows represent the set of available coins. The first row is for the empty set of coins, the second for {1c}, the third for {1c, 3c}, and the fourth for {1c, 3c, 4c}. The values in the table represent the minimum number of coins needed to make the amount. Green arrows show the recurrence relation: from (i, m) to (i, m - v_i) and from (i, m) to (i - 1, m). A red arrow points from the value 2 in the cell (1c, 3c, 4c, 6) to the value 2 in the cell (1c, 3c, 4c, 2), indicating that the optimal solution for 6 units is to use a 4c coin and a 2c coin (which is the optimal solution for 2 units).

Lessons on dynamic programming

- It is necessary to study each problem on a case-by-case basis.
- Storing a large number of partial results, which requires significant memory usage.
- Suitable for only certain problems (min, max, counting the number of solutions).